

Quadratic equation

Question

If $y = \frac{x^2 + 2x + \lambda}{2x - 3}$, ($x \neq -\frac{3}{2}$) and x is real, find the possible range of values of λ for which y takes all real values.

Solution

$$y = \frac{x^2 + 2x + \lambda}{2x - 3}, \quad (x \neq -\frac{3}{2}) \quad (1)$$

$$\begin{aligned} \therefore (2x - 3)y &= x^2 + 2x + \lambda \\ x^2 + (2 - 2y)x + (\lambda + 3y) &= 0 \end{aligned} \quad (2)$$

Since x is real, Δ of (2) ≥ 0 .

$$\begin{aligned} (2 - 2y)^2 - 4(\lambda + 3y) &\geq 0 \\ (1 - y)^2 - (\lambda + 3y) &\geq 0 \\ y^2 - 5y + (1 - \lambda) &\geq 0 \end{aligned} \quad (3)$$

<u>Method 1</u>	<u>Method 2</u>
Let $t = y^2 - 5y + (1 - \lambda)$ Since y takes up all values of real values and t is a quadratic function in y, from (3) the graph of t cannot cut the y-axis in the y-t plane. $\therefore \Delta$ of (3) ≤ 0 $(-5)^2 - 4(1 - \lambda) \leq 0$ $\therefore \lambda \leq -\frac{21}{4}$ (4)	From (3), completing square, $\left(y - \frac{5}{2}\right)^2 + \left(-\lambda - \frac{21}{4}\right) \geq 0$ (*) Since y takes up all real values, (*) is always true if $-\lambda - \frac{21}{4} \geq 0$ $\therefore \lambda \leq -\frac{21}{4}$ (4)

However, if $\lambda = -\frac{21}{4}$,

$$y = \frac{x^2 + 2x - \frac{21}{4}}{2x - 3} = \frac{4x^2 + 8x - 21}{4(2x - 3)} = \frac{(2x - 3)(2x + 7)}{4(2x - 3)} = \frac{1}{4}(2x + 7), \quad (x \neq -\frac{3}{2})$$

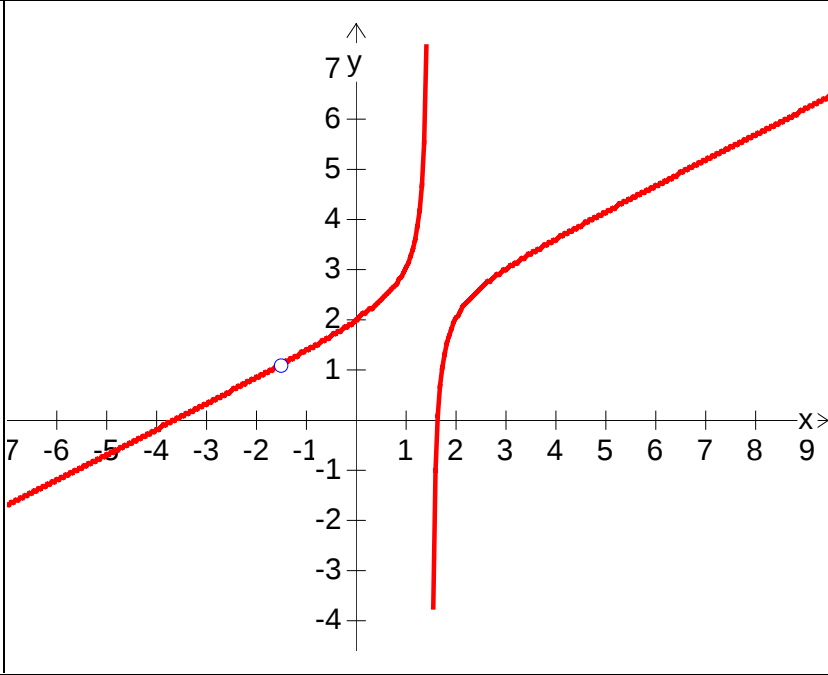
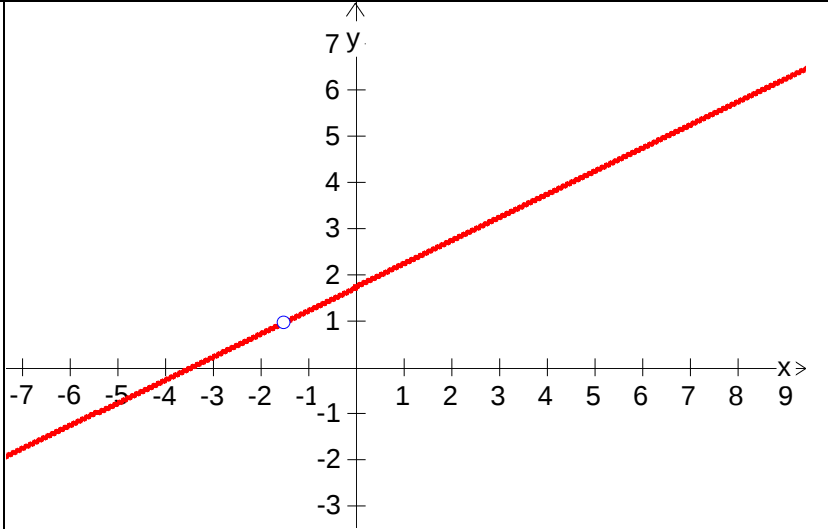
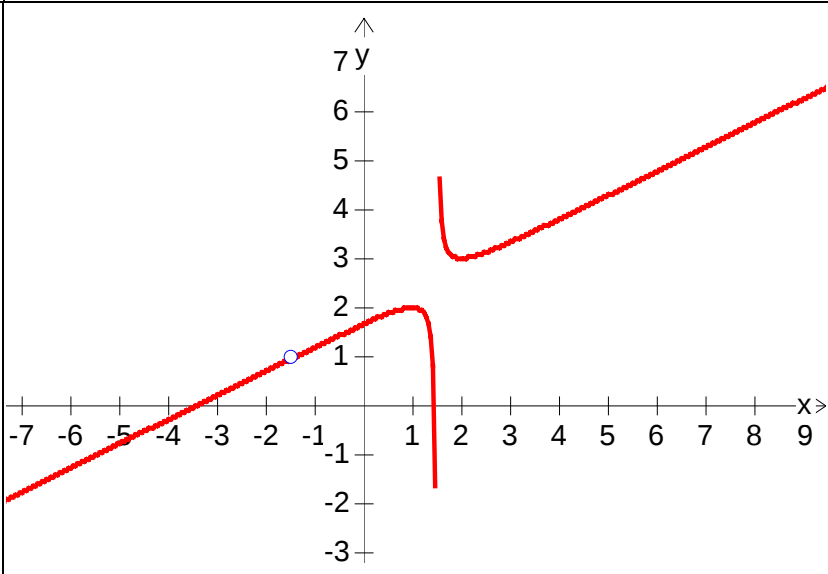
Since $x \neq -\frac{3}{2}$, $y \neq \frac{1}{4}\left[2\left(-\frac{3}{2}\right) + 7\right] = 1$, y can take up all real values except $y = 1$.

This gives a straight line with a hole.

$\therefore \lambda = -\frac{21}{4}$ is rejected.

$\therefore$ The final solution is $\lambda < -\frac{21}{4}$.

The graphs with different λ will illustrate the point.

$\lambda < -\frac{21}{4}$ y can takes up all real values. x is real , $x \neq -\frac{3}{2}$. The graph on RHS takes: $\lambda = -6$	 The graph shows a red curve on a Cartesian coordinate system. The x-axis ranges from -7 to 9, and the y-axis ranges from -4 to 7. The curve consists of two branches. The left branch starts from the bottom left, passes through a blue open circle at $(-1.5, 1)$, and continues upwards and to the right. The right branch starts from the bottom right, passes through the same blue open circle, and continues upwards and to the right. There is a vertical asymptote at $x = 1.5$.
$\lambda = -\frac{21}{4}$ This gives a straight line with a point $\left(-\frac{3}{2}, 1\right)$ missing. y cannot take up all real values.	 The graph shows a straight red line on a Cartesian coordinate system. The x-axis ranges from -7 to 9, and the y-axis ranges from -3 to 7. The line passes through a blue open circle at $(-1.5, 1)$. There is a jump discontinuity at $x = -1.5$, where the point $(-1.5, 1)$ is missing.
$\lambda > -\frac{21}{4}$ The graph on RHS takes: $\lambda = -4$ Note that y cannot take up all real values, for example, check $y = 2.5$ in the graph.	 The graph shows a red curve on a Cartesian coordinate system. The x-axis ranges from -7 to 9, and the y-axis ranges from -3 to 7. The curve consists of two branches. The left branch starts from the bottom left, passes through a blue open circle at $(-1.5, 1)$, and continues upwards and to the right. The right branch starts from the bottom right, passes through the same blue open circle, and continues upwards and to the right. There is a vertical asymptote at $x = 1.5$.