Quadratic equation

Question

If $y = \frac{x^2 + 2x + \lambda}{2x - 3}$, $(x \neq -\frac{3}{2})$ and x is real, find the possible range of values of λ for which y takes **all** real values.

Solution

$$y = \frac{x^2 + 2x + \lambda}{2x - 3}, \quad (x \neq -\frac{3}{2}) \qquad \textbf{(1)}$$

$\therefore$ $(2x – 3)y = x^2 + 2x + \lambda$

$x^2 + (2 – 2y)x + (\lambda + 3y) = 0$ **(2)**

Since x is real, Δ of **(2)** ≥ 0.

$(2 – 2y)^2 – 4(\lambda + 3y) \geq 0$

$(1 – y)^2 – (\lambda + 3y) \geq 0$

$y^2 – 5y + (1 - \lambda) \geq 0$ **(3)**

Method 1	Method 2
Let $t = y^2 – 5y + (1 - \lambda)$ Since y takes up all values of real values and t is a quadratic function in y, from **(3)** the graph of t cannot cut the y-axis in the y-t plane. $\therefore$ Δ of **(3)** ≤ 0 $(-5)^2 – 4(1 - \lambda) \leq 0$ $\therefore$ $\lambda \leq -\frac{21}{4}$ **(4)**	From (3), completing square, $\left(y - \frac{5}{2}\right)^2 + \left(-\lambda - \frac{21}{4}\right) \geq 0$ **(*)** Since y takes up all real values, **(*)** is always true if $-\lambda - \frac{21}{4} \geq 0$ $\therefore$ $\lambda \leq -\frac{21}{4}$ **(4)**

However, if $\lambda = -\frac{21}{4}$,

$$y = \frac{x^2 + 2x - \frac{21}{4}}{2x - 3} = \frac{4x^2 + 8x - 21}{4(2x - 3)} = \frac{(2x - 3)(2x + 7)}{4(2x - 3)} = \frac{1}{4}(2x + 7), \quad (x \neq -\frac{3}{2})$$

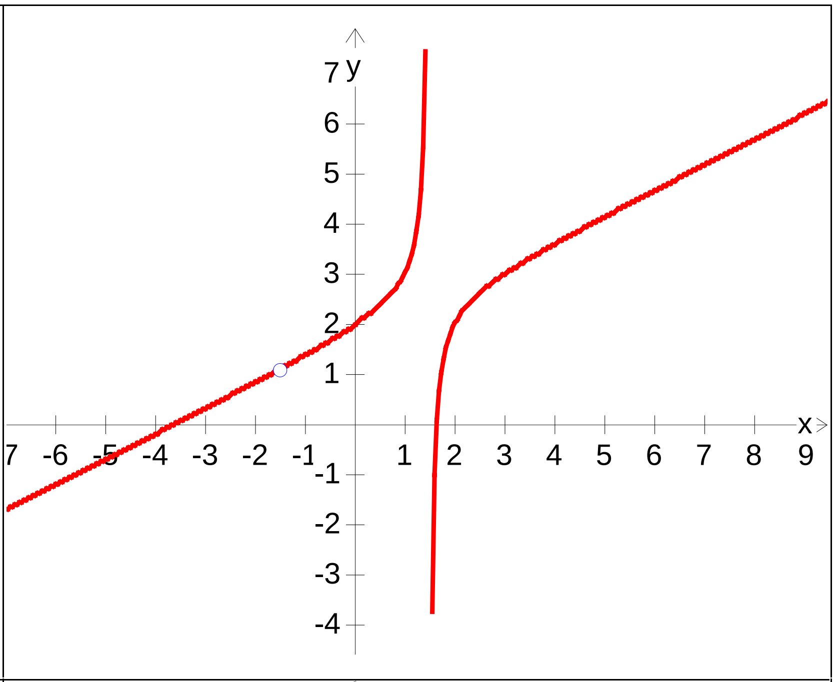
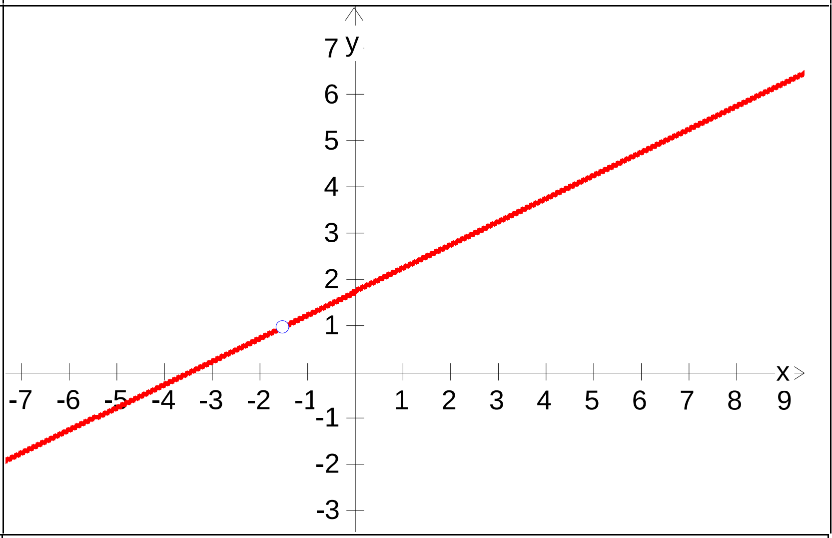
Since $x \neq -\frac{3}{2}, y \neq \frac{1}{4}\left[2\left(-\frac{3}{2}\right) + 7\right] = 1$, y can take up all real values except y = 1.

This gives a straight line with a hole.

$\therefore$ $\lambda = -\frac{21}{4}$ is rejected.

$\therefore$ The final solution is $\lambda < -\frac{21}{4}$.

The graphs with different λ will illustrate the point.

$\lambda < -\frac{21}{4}$ y can takes up all real values. x is real , $x \neq -\frac{3}{2}$. The graph on RHS takes: λ = -6	
$\lambda = -\frac{21}{4}$ This gives a straight line with a point $\left(-\frac{3}{2}, 1\right)$ missing. y cannot take up all real values.	
$\lambda > -\frac{21}{4}$ The graph on RHS takes: λ = -4 Note that y cannot take up all real values, for example, check y = 2.5 in the graph.